LIMITATIONS AND THE WORLD BEYOND

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ABSTRACT: This paper surveys our inescapable limits as cognitive agents with regard to a full world of fact: the well-known metamathematical limits of axiomatic systems, limitations of explanation that doom a principle of sufficient reason, limitations of expression across all possible languages, and a simple but powerful argument regarding the limits of conceivability. In ways demonstrable even from within our limits, the full world of fact is inescapably beyond us. Here we propose that there must nonetheless be a totality of fact, and that despite our limits we can know something of its general character. The world as the totality of fact must form a plenum, with a radically unfamiliar formal structure that contains distinct elements corresponding to each element of its own power set.

KEYWORDS: truth, language, Georg Cantor, Kurt Gödel, totality, plena

1. Introduction

Our topic is that of limits: the metamathematical limits of axiomatic systems, epistemic limits of explanation, linguistic limitations of expression, conceptual limits of conceivability, and ultimately questions of ontological and metaphysical limits as well. The limitations of axiomatic demonstration and of mechanical computation are clear from the Turing and Gödelian traditions. In section 2 we pursue extensions and analogies to limitations intrinsic in the structure of explanation, restrictive on a principle of sufficient reason PSR. In section 3 we consider the limitations on expression entailed by recursive linguistic structure, extending the argument from single languages to sets of possible languages and showing that even the properties of languages inevitably outstrip the properties expressible within those languages. In Section 4 we pause to consider epistemic implications, extending the discussion beyond language to incompleteness of any body of *conceivable* truths in the face of a demonstrably larger realm of fact. We suggest nevertheless that something can be shown of its general character. The world as the totality of fact must form a plenum,[1] with implications we here set out

[1] Nicholas Rescher and Patrick Grim, "Plenum Theory," *Noûs* 42 (2008): 422, *Beyond Sets: A Venture in Collection-Theoretic Revisionism* (Frankfurt: Ontos Verlag, 2011).

to explore.

Plato's *Timaeus* launched the pivotal belief of ancient Neo-Platonism that Reality reflects the operations of Reason and accordingly constitutes a rationally intelligible manifold. In consequence man, the rational animal, is able to get a reason-engendered cognitive grip on Reality's key features. This fundamental idea was to become one of the mainstays of Western philosophy. But no-one, then or since, maintained that human reason's grip on Reality was complete or completable--that human cognition and speculation could exhaust the unbounded vastness of possibility and plumb the bottomless depths of its relationship to the real--a task which, if achievable at all, required an intelligence of supra- and super-human capacity. But just where can we find clear signs of the limits of human intellection and pinpoint some of the issues that lie beyond the horizons of our cognitive reach. No doubt this is a difficult question but there are some things that can plausibly be said on the problem and hopefully some of them will be said here.

The limitations we track are characteristically not some boundary imposed from without but intrinsic limitations of reach from within an entire method of axiomatization, explanation, expression, or comprehension. The problematic clearly traces to Kant for whom human cognition has limits by way of limitations (Grenzen) but not boundaries (Schranken), there being no wall or fence that somehow ontologizes those limits. For us those limits lie not as with Kant, in the faculty structure of the human intellect, but in the nature of the conceptual resources characteristic of our cognition, or perhaps of any cognition.

In sections 5 and 6 we attempt to go farther metaphysically and ontologically, for a glimpse of the world beyond our limits. The attempt itself sounds paradoxical, and it is in fact paradox that we take as the key. The world as the totality of fact lies inevitably beyond our limitations—explanatory, expressive, and conceptual. But we propose we can nonetheless know something of its general character. The world as the totality of fact must form a *plenum*.

2. Limits from Axiomatization to Explanation

The limitations of axiomatization are well known. No formal system adequate for basic arithmetic can be both consistent and complete. No axiomatic system can contain as theorems both all and only the truths expressible in the formal language of the system. We cannot hope to grasp all of mathematical truth—restricted even to the mathematical truth we have the means to express—with the techniques of axiomatization.

It is a short step from Gödel to Turing, from formal systems to mechanical algorithms. By the same token, and in much the same way, no mechanical algorithm can give us all and only correct answers to some easily expressible questions about the function of mechanical algorithms. In both the Gödel and Turing results, it is the system itself—by a particular power of embedding—that reveals its own limitations. It is because a system for number theory can represent (or echo) any mechanism of axiomatic deduction that any axiomatic system will be provably incomplete. It is because Turing machines can echo and embed any algorithmic mechanism that there can be no faultless algorithmic mechanism for any of a range of basic questions regarding them all.

We will return to explanation and the Principle of Sufficient Reason (PSR) at a number of points. Here we start with a particularly simple version:

> (PSR-T) For every truth there is some other, epistemically distinct truth that provides a cogent explanation for it.

If we take 'explanation' to demand a deductively valid accounting, PSR-T will be untenable for precisely Gödelian reasons. Any deductive system adequate for scientific explanation will have to be adequate for arithmetic. But any deductive system adequate for arithmetic, there will be truths expressible in the system which will not be deducible as theorems. Those will be truths in violation of PSR-T.

We can take the result further, and make it more pressing, by *replacing* the concept of deduction in the Gödel result with a concept of explanation instead. Mathematical exploration through the last century, eloquently expressed in Hilbert, was a vision of some distant but reachably complete completed mathematics. That vision died with Gödel's proof. A vision of a completed explanatory *science* has spurred scientific exploration in much the same way. That vision of scientific explanation is as impossible as the correlate vision of mathematical explanation, and for precisely the same reasons.

Suppose a science which contain (a) a complete set of basic facts, and (b) a complete set of 'explanatory consequence' principles whereby further facts follow from others. It is clear that any such system must also contain the mechanisms of any system adequate for arithmetic. Among its 'basic facts' must be the axioms and among its 'explanatory consequences' principles must be the rules of inference which are required for basic arithmetic. It then follows that there will be true statements in the language of such a science for which our 'completed science' will be unable to offer a scientific explanation.

There is an older and simpler problem with PSR-T, of course. The explanatory project confronts us with the prospect of basic explanatory elements analogous to axioms which by hypothesis cannot be derived from anything else. Further forms of the principle of sufficient reason, correlate to even wider limitations on explanation, reappear later in our discussion.

3. Intrinsic Limits of Language and Truth

We humans conduct our cognitive business by means of language, broadly conceived to include all processes of symbolic communication. Linguistic articulation, both in human communicative reality and in its formal representation, is fundamentally recursive. Beginning with a finite vocabulary it exfoliates meaningful statements by means of a finite number of grammatical rules of combination. The result is a potentially infinite number of meaningful statements in in any such language, but those statements will be enumerable and thereby denumerable in number. And of course if the meaningful statements (the well-formed formulas as a whole) can be enumerated (and thus be denumerable in number) this will also have to hold for the subset of them that are true. The truths expressible in any language, in sum, form a denumerable manifold.

At this point a distinction between truths and facts becomes critical. We take truths to be linguistically articulated claims—specifically those that are correct. We take facts to be something else again: states of affairs that obtain and do so independently of any articulation by linguistic means.

A. We begin with the simplest formal case, which is also closest to the reality of human languages. Consider a language with a finite number of basic symbols and a finite number of recursive rules for combination. Such a language will afford us with a countably infinite number of formulae. At best, the expressible truths for such a language will be countably infinite.

It's clear that there will be more than countably infinite facts, a point provable using the example of this language alone. The formulae of any such language *L* form a countably infinite set. But by the basic mechanisms of Cantor's Theorem, there will be more elements of the power set of any set than elements of that set itself. Consider then the power set *PL* of the set of formulae of this initial language. For each set element of *PL* there will be a distinct fact: the fact that a specific formulae does or does not belong to that set, for example. Even this small corner of a world of fact—facts about the language *L*—will have facts inexpressible in *L* itself. The facts *about* such a language inevitably outstrip the truths it can express.

What are we to make of there being infinitely more actual facts than articulable truths? With human knowledge functioning linguistically by way of a recognition and acknowledgement of truths, does this disparity between facts and truths not entail the existence of an unknowable truth?

Here it is instructive to begin with a simple analogy: that of Musical Chairs.

Where there are more players than chairs it is inevitable that some will be left unseated when the music stops. So the existence of *unseated* players is inescapable. But this of course does not itself mean that any players are *unseatable* so that it is in principle impossible for such a player to be seated. The prospect of seating cannot be denied to any of them.[2] When this situation is analogized to the truth/fact situation, we will have it that the inevitability of *unknown* facts does not of itself establish the existence of *unknowable* ones. All we can maintain at this point is that there are bound to be *unknown* facts: that there are unknowable ones does not follow. That *not every fact can be known* does not of itself enjoin that *some fact cannot possibly be known*. The quantitative disparity between formulable truths and objective facts does not immediately establish the existence of unknowable facts.

B. What of the truths expressible by any *possible* language of this simple formal and very human form, involving finitely many basic symbols and finitely many recursive rules of combination? We begin by supposing that each possible language takes its basic symbols from some zingle but countably infinite reservoir of possible symbols, awash with as many basic symbols as there are numbers 1, 2, 3… On that assumption, the basic symbol sets of the full set of our possible languages will be enumerable: there will be only a countably infinite number of basic symbol sets.

Because those finite sets of symbols can simply be appended as the first of the countably infinite formulae generable using them, within our basic assumptions we can envisage an enumeration of all formulae of all possible languages of this form as an infinite series of infinite arrays. Using $s1_{L1}$ through sn_{L1} to represent the finitely many basic symbols of language 1 and $f1_{L1}$, $f2_{L1}$, $f3_{L1}$… to represent its infinitely many compound formulae, such an array might take this form:

Language 1	$s3_{L1}$, $s4_{L1}$, $s3_{L1}$, $s4_{L1}$ … sn_{L1}, $f1_{L1}$, $f2_{L1}$, $f3_{L1}$, $f4_{L1}$, …
Language 2	$s1_{L2}$, $s2_{L2}$, $s3_{L2}$, $s4_{L2}$ … sn_{L2}, $f1_{L2}$, $f2_{L2}$, $f3_{L1}$, $f4_{L2}$, …
Language 3	$s1_{L3}$, $s2_{L3}$, $s3_{L3}$, $s4_{L3}$ … sn_{L3}, $f1_{L3}$, $f2_{L3}$, $f3_{L3}$, $f4_{L3}$, …
Language 4	$s1_{L4}$, $s2_{L4}$, $s3_{L4}$, $s4_{L4}$ … sn_{L4}, $f1_{L4}$, $f2_{L4}$, $f3_{L4}$, $f4_{L4}$, …

As in Cantor's proof for the countability of the rationals, however, we can introduce a circuitous but systematic enumeration of every item in that array as well:

[2] In logical notation, the different at issue is that between $\boxtimes(\exists x){\sim}Sx$ and $(\exists x)\boxtimes{\sim}Sx$.

Language 1	$s3_{L1}$, $s4_{L1}$, $s3_{L1}$, $s4_{L1}$ … sn_{L1}, $f1_{L1}$, $f2_{L1}$, $f3_{L1}$, $f4_{L1}$, …
Language 2	$s1_{L2}$, $s2_{L2}$, $s3_{L2}$, $s4_{L2}$ … sn_{L2}, $f1_{L2}$, $f2_{L2}$, $f3_{L1}$, $f4_{L2}$, …
Language 3	$s1_{L3}$, $s2_{L3}$, $s3_{L3}$, $s4_{L3}$ … sn_{L3}, $f1_{L3}$, $f2_{L3}$, $f3_{L3}$, $f4_{L3}$, …
Language 4	$s1_{L4}$, $s2_{L4}$, $s3_{L4}$, $s4_{L4}$ … sn_{L4}, $f1_{L4}$, $f2_{L4}$, $f3_{L4}$, $f4_{L4}$, …

On the assumption of a countable reservoir of basic symbols, then, there will be only countably many truths expressible in all *possible* languages of this basic form. We know the facts of even one of those languages form more than a countable set, and thus the facts regarding even one of these possible languages outstrip the truths expressible in all such possible languages.

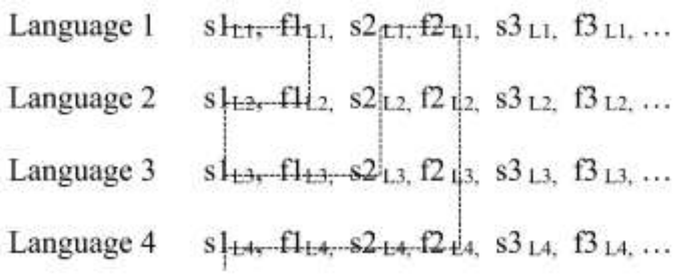
C. But perhaps we've sold linguistic possibilities short. We can expand our conception of formal languages, recognizing as we do so that we are leaving the limitations of human languages behind.

Limitations like those above are demonstrable for even some superhuman languages. Let us start by allowing a language to contain more than a finite number of basic symbols. It is indeed standard in outlining formal systems to envisage a countably infinite number of basic formulae p1, p2, p3…. That change alone won't alter the results for single languages. The countably infinite basic symbols of such a language can be interwoven with the countably infinite formulae that can be recursively generated from those formulae, giving us no more than countably infinite formulae over all. The cardinality of our formulae, the factual limitations of truths, will remain.

As long as our basic symbols are drawn from a countably infinite pool, the same will hold for all *possible* languages of such a form. For each language we can envisage an enumeration that interweaves the countable series of basic symbols with the countable series of recursively combinatorial formulae:

Language 1	$s1_{L1}$, $f1_{L1}$, $s2_{L1}$, $f2_{L1}$, $s3_{L1}$, $f3_{L1}$, …
Language 2	$s1_{L2}$, $f1_{L2}$, $s2_{L2}$, $f2_{L2}$, $s3_{L2}$, $f3_{L2}$, …
Language 3	$s1_{L3}$, $f1_{L3}$, $s2_{L3}$, $f2_{L3}$, $s3_{L3}$, $f3_{L3}$, …
Language 4	$s1_{L4}$, $f1_{L4}$, $s2_{L4}$, $f2_{L4}$, $s3_{L4}$, $f3_{L4}$, …

All formulae in all languages can be enumerated as before:



The formulae of all possible languages based on countably infinite symbols from a countably infinite pool will still form merely a countable set. The truths expressible in all possible languages of such a form will be merely countable.

D. The situation changes if we further broaden assumptions, leaving human capabilities even farther behind. Consider the possibility of a larger reservoir from which a language might draw its basic symbols: a reservoir that has as many basic symbols not merely as the rationals, for example, but as many as the reals.

Any language that has either a finite number of basic symbols drawn from such a pool or a countably infinite number of such symbols will be limited, as above, to a countably infinite number of formulae. But the conclusions drawn so far will not hold for all *possible* languages of this expanded form. A very simple way of seeing this is to envisage those languages that have merely one basic symbol. Since that symbol can be any of a collection as large as the reals, we will not be able to enumerate all of those languages, prohibiting the countable list of languages used on the left axis in the arrays above. For languages with basic symbols drawn from a set the size of the reals, then, formulae of *each* language will be countable but formulae of all *possible* such languages will not.

Limitations of countably many formulae are obviously lifted for even *single* languages if we allow a language to have as many simple formulae as the reals. Somewhat less obviously, limitation to the countably infinite is lifted for a single language with countably many basic formulae and infinite combinations: infinite conjunctions or disjunctions, for example. We might list conjunctions in such a language by using 0 or 1 to indicate whether they include symbol 1, symbol 2, symbol 3, and so on:

conjunction contains:	s1	s2	s3	s4	s5	s6	s7	s8 ....
conjunction 1 :	0	1	1	0	1	0	0	1 . . .
conjunction 2:	1	0	0	0	1	0	1	0 . . .
conjunction 3:	0	0	0	1	0	1	0	0 . . .

It is clear that every infinite series of 1's and 0's will be represented by some conjunction in such a system. But these correspond to the infinite decimals between 0 and 1, which correspond to the reals. Cantor's proof that there are non-denumerably many reals may be performed quite directly on any proposed enumeration of these conjunctions. We can produce a conjunction *not* on the list by exchanging 1's and 0's on the diagonal.

E. If we weaken assumptions and stretch possibilities for languages far enough, then, we can have sets of possible languages and even single languages that transcend the limits of a countable infinity of expressible truths. In a very real sense, however, such languages bring us no closer to the world of facts.

No matter how large the set of formulae expressible in any of these languages, the power set of that set will be larger than the set itself. For every set element of that power set there will be a fact: the fact that a given formulae is or is not an element of that set, for example. There will still be more facts expressible in any given language.

Given any set of specifications for a form of language, there will be a set of formulae and thus a set of truths expressible in all *possible* languages of that form. The power set of *that* set of all possible formulae or expressible truths will be larger still, and thus the facts even about sets of truths expressible in all possible languages of a specific form will transcend the truths so expressible. Like the individual languages within them, ranges of possible languages embody more facts than they can possibly express.

F. All of the arguments presented to this point have been written in terms of syntax: numbers of formulae generable within a given language. But languages in the sense we are after are perhaps better conceived of semantically, such that formulae are *about* certain things, using predicates to express properties *of* certain things. A more semantic and in that sense more philosophical form of the argument makes the point in its most general form.

With any language there will be those things that it can say things about: what we might term the *linguistic objects* of a language. Things in general, linguistically reachable within such a language or not, we can term *factual objects*. Any language will also have those things that it can say *about* things: its *predicates*. Factual properties that actually hold of things, linguistically bound or not, we will simply term *properties*.

On this simple outline, it's clear that the predicates of any language will themselves be factual objects. By an analog of Cantor's Theorem, we know that the sets of those objects outnumber the objects themselves. But for each such set, there is a unique property, indeed an extensional property: the property of belonging to such a set, for example. There are therefore more properties of factual objects than there are predicates available in any language to express those properties. Indeed there are more properties of the predicates of any language than there are predicates in the language. The facts of properties inevitably outstrip truths expressible by predicates.

What holds for a single language holds for all possible languages. If we consider the predicates applicable in any possible language, of whatever form, we are considering a set of factual objects. But there will be more sets of such objects, and thus there will be more factual properties, than there are predicates applicable in any possible language.

G. Does this entail that there is any specific inexpressible truth? One can hardly ask for an example. To this point, considering languages both syntactically and semantically, the image of musical chairs still holds: each language will leave out some fact, but nothing yet identifies a specific fact that will be left out.

Languages are more than syntactic structures, more even than syntactic structures with correspondences to objects and properties. Languages are means of managing information. Information is packaged in the form of expressions, unpacked by means of derivation. It is in terms of information that we can begin to see some specifics regarding linguistic limits: for any language L, a *specific* body of information beyond it.

We have termed truths those linguistic elements that correspond to facts. For any language there will be those truths expressible in the language. Each truth will embody some information, reflecting some fact. But there is one body of information that will inevitably escape a language, in one way or another: that body of information that is represented in *all* of its truths combined. For any language L, we will term that megafact M_L. There is no single truth in L that can capture this megafact: totalistic self-representation cannot be internalized declaratively.

Suppose any language L, and all truths expressible in L. Consider moreover a truth-preserving set of rules of derivation R employed in L which allows one to squeeze out as consequent truths the information contained in a given truth. Finally, consider M_L, the information contained in all the truths of L. M_L will either be L-inexpressible or R-inaccessible at least in part. M_L will be inexpressible in L in any way in which all the information in M_L will be derivable by R.

Our discussion started with Gödel and Turing as a foundation, moving from there to considerations that were largely Cantorian. Here the argument turns on Gödel once again. Were M_L both L-expressible and fully R-accessible, there would be an axiomatic system with R as its rules and M_L as an axiom from which all truths expressible in L were derivable. By Gödel, there can be no such axiom system.

The result will clearly hold for all the kinds of languages to which Gödel applies: all those satisfying the minimal requirements of an L and R adequate for arithmetic. It is also possible to generalize the result beyond those specific requirements.[3] Given any rules of derivation R, a language that can represent R-derivability we will call R-expressive. A language that can take any of its own expressions as object we will term expressibility-reflective. For any expressibility-reflective language L that is R-expressive, for any truth-preserving R, the megafact M_L for that language will either be inexpressible in L or R-inaccessible at least in part: either M_L will be inexpressible in the language of L, or there will be information in M_L that will be underivable by R.

For any language within these minimal constraints there will be a particular fact that proves inaccessible for it: the megafact M_L that represents the totality of information in the facts that it does represent. Note that M_L doesn't have to extend to all facts. It is specified relative to a language and encapsulates merely the information expressible in the facts captured in that language. Even that smaller language-relative totality of facts escapes the nets of language and derivability.

H. Here again our reflections impact the principle of sufficient reason.

Anything rationale offered as an explanation, in any language, will be a set of expressions within that language. The available rationales for any language will therefore be limited by the available expressions. For a standard language L with countably many expressions, for example, there will be only countably many possible finite rationales.

It's clear from the pattern of argument above that for any L there will be not only more facts than linguistically expressible truths, but more facts than there are available rationales. Using 'explanation_L' to indicate rationales in language L, then, the following version of the PSR will fail for any L:

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[3] Here the generalization of Gödel follows roughly the lines of chapter 3 of Patrick Grim, *The Incomplete Universe* (Cambridge, MA: MIT Press, 1991).

(PSR-F) Every distinct fact has a distinct explanation$_L$.

The lesson will extend to the languages of non-standard forms considered above. It will also extend to explanation in any or all possible languages. If we consider the rationales expressible in any possible language, of whatever form, we are considering a set of factual objects. But there will be more sets of rationales than rationales themselves. For each of those sets there will be a distinct fact. There will therefore be more distinct facts than distinct rationales in any possible language. Generalizing 'explanation' from 'explanation in L' to 'explanation in *any* possible language,' this more encompassing version of the PSR will fail as well:

(PSR'-F) Every distinct fact has a distinct explanation.

4. Epistemic Reflections and Conceivability

What does this disparity between linguistic truth and trans-linguistic fact mean for our knowledge? To what extent do the limitations of language extend to limits of conceivability?

A. At first glance, axiomatization as a model of a distinction a distinction between explicit and implicit knowledge might seem to offer some hope.

By the Cantorian argument, the expressible truths of any language will be outnumbered by the facts. But there are two ways of affirming or claiming a fact. One is to state it explicitly and specifically, in the form for example of a corresponding truth. Another is to affirm it obliquely and implicitly by stating other facts from which it follows. In that sense a single statement---the conjunction of the axioms of a system, for example—can be seen as implicitly containing the full information of all theorems of the system. It lies in the logic of things that one truth can informatively encompass a vast—indeed a potentially infinite—multitude of other distinct claims.

One true claim, such as a conjunction of the axioms of plane geometry, can informationally encompass the entire field. Finite access to claims does not itself therefore entail finitude in knowledge. Given the distinction between explicit expression and implicit deducibility on the model of axioms, the quantitative disparity between truth and fact might not seem all that portentious.

We might, then, distinguish two basic questions:

Q1. Can the totality of the facts in the domain at issue be stated and acknowledged explicitly in terms of coordinate truths?

Q2. Can the totality of fact of the domain at issue be substantiated at least obliquely and implicitly by way of inferential axiomatization?

The force of the Cantorian argument—there are more facts than truths with which to express them—is that the answer to Q1 is a clear 'No.' But for standard systems, at least, a Cantorian argument shows that the answer to Q2 must be 'No' as well.

Standard systems will have only a countable number of theorems. Even implicitly, therefore, their axioms will contain only a countable number of truths. Implicit knowledge amounts to deductive closure: we implicitly know whatever can be derived from what we explicitly know. Derivation is a recursive process. It begins with premisses and applies stepwise any of a finite register of inferential rules. A body of explicit axioms, then, be it finite or countably infinite, can never represent more than a countable body of implicit knowledge. In the previous section we envisaged systems beyond standard systems. But even these will have only some limited cardinality of implicit theorems—a cardinality that will be provably exceeded by the range of fact…even the range of fact about those theorems.

If our model of implicit knowledge is axiomatic, it must be recognized that the power of an axiomatic system cannot exceed that of the language in which it is expressed. Our results above hold for all languages, and thus for the implicit knowledge contained in any axioms written within those languages as well. Any hope for conceivability beyond linguistic limits must appeal to something beyond implicit knowledge, at least implicit knowledge conceived on the model of axiomatization.

B. Given the distinction between facts and linguistic truths employed throughout, there is another question close to that above. Here the question is again one of implicit as opposed to explicit knowledge, but limited merely to the facts expressible in a language:

> Q3. Can the totality of *truth* in the domain at issue be claimed and affirmed at least obliquely and implicitly on the model of inference from axioms?

This question demands something more like a Gödelian than a Cantorian analysis. Here again, in ways allied with considerations above, the answer will be 'no.'

For any system adequate for arithmetic, and therefore of course for realms of truth and fact at large, there will be *truths* expressible in the language that are not deducible from the axioms. If even expressible truths within a language outstrip the implicit information of any axiom set, the implicit knowledge contained in axioms does not seem to offer an escape.

The question of implicitly knowing M_L is particularly instructive. M_L is too 'large' to be seen as a consequence of some other truth in the system: it contains by definition all information of all truths in the system. Nor can it function as an axiom which implicitly contains all other information, as long as 'implicitly' is taken on the model of inference from a consequence function R. By the results above, no system can express an M_L from which all information is recoverable by inference.

Any appeal to implicit knowledge in the hopes of overcoming the limits we've documented above must appeal to implicit knowledge conceived on some model other than that of axiomatic containment or logical inference. The distinction between implicit and explicit knowledge remains an intriguing one, one that will reoccur in thinking about conceivability and reference to a world beyond.

C. Does essential limitation of knowledge doom us to error?

The numerical discrepancy between truth and fact means that our knowledge of a world of fact is bound to be imperfect. Specifically it means this knowledge is incomplete. Does it also mean that it is incorrect—that it contains not only gaps but errors? After all, suppose that you are otherwise fully informed about swans in general but totally unaware the some Australian Swans are black. One is then bound to arrive at the erroneous conclusion that all swans are white.

The incompleteness of our knowledge does not, of course, *ensure* its incorrectness—after all, even a single isolated belief can represent a truth. But it does strongly *invite* it. For if our information about some object is incomplete then it is bound to be unrepresentative of the objective make up-as-a-whole so that a judgment regarding that object is liable to be false. The situation is akin to that depicted in John Godfrey Saxe's "The Blind Men and the Elephant" which tells the story of certain blind sages who variously read incomplete evidence as indicating a creature like a wall, like a spear, a snake, a fan, or a rope. "Each was partly right," Saxe concludes, "And all were in the wrong."

The lesson is clear. The incompleteness of object-descriptive statements certainly does not entail their incorrectness: incomplete information does not ensure false belief with categorical necessity. But it does ensure inadequate understanding since at the level of generality there will be too many gaps that need filling in. There are just too many alternative ways in which reality can round out an incomplete account to warrant confidence in the exclusion of error.

This vulnerability of our putative knowledge of the world in the face of potential error is rather *exhibited* than *refuted* in our scientific knowledge. For this is by no means as secure and absolute as we like to think. We cannot but recognize in our heart of hearts that our putative truth in fact incorporates a great deal of error. There is every reason to believe that where scientific knowledge is concerned further knowledge does not just supplement but generally corrects our knowledge-in-hand, so that the incompleteness of our information implies its presumptive incorrectness as well.

D. To this point we have concentrated on the disparity between the limited world of linguistic truth and the larger world of fact beyond, but the range of these deliberations can be extended yet further. It is not merely in language that we manage our attempts at grasping facts, but in conceptualization and thought. Although neither speculation nor conceptualization need be recursively conceived or recursively limited, the same quantitative disparity between epistemic thinkability and ontological actuality will obtain in these contexts as well.

Is there reason to think that the realm of fact must outstrip pure conceivability? We have seen the limitations of language, and a long philosophical tradition insists that the limitations of language are necessarily the limitations of conceivability and therefore of knowledge as well. If we conduct the business of conception and knowledge via language, the limitations we've already noted, essential to any language, will be limitations of conceivability and knowability as well.

But limitations will still face us even if we abandon the assumption that conceivability and knowledge are tied to language. Let us assume a notion of conceivable propositions beyond the limits of linguistic expression: the conceptual parallel to facts rather than truths, perhaps. Consider all the propositions you have entertained in the course of reading this article, or all the propositions that have come to mind throughout the day. Consider all the propositions you have ever entertained, or all the propositions which you will in fact entertain throughout your lifetime.

The world of fact will necessarily outstrip any such set of propositions. There will be more subsets of propositions than there are propositions themselves. For each of these, there will be a specific fact: that a given proposition P is or is not a member of that set, for example. There will then be more factual propositions than those that you conceive in a day or indeed that all humans conceive in the course of human history. The world of fact will necessarily outstrip the realm of propositions conceived, and thus of course of things known.

The argument takes us even further. For consider not merely the propositions that have or will be conceived, but the propositions it is in any way possible to conceive: not merely the conceived but the *conceivable* propositions. For even these a numerical argument will apply: there will be more subsets of propositions, and thus more facts, than there are *conceivable* propositions.

The implication is that there are facts that are not even *conceivable*. That conclusion, of course, is one that holds on the level of generality. We cannot meaningfully claim to know—or even conceive—of any of them. The claim that there are inconceivable facts is in that regard like the claim that there are facts that I do not in fact know. I can conceive of there being inconceivable facts, of course without being able to conceive of any of the specifics, just as I can know there are facts I don't in fact know, of course without knowing any of those specific facts.

Unlike the image of musical chairs, the inconceivable facts would have to be specific inconceivable propositions. The realm of what is actually conceived, by a person on a day, in a lifetime, across all human history or by all creatures capable of entertaining propositions might have been different. But the realm of what is conceiv*able* in any of these categories would seem to be metaphysically fixed. If there are more facts than there are conceivable propositions, there must be *specific* facts beyond the range of propositional conceivability.

E. There is an air of paradox at this point: in conceiving of inconceivable facts, have we not somehow made them conceivable after all? Hints of paradox do mark any attempt to glimpse the world beyond, but there are several relevant considerations here.

Here as before we might appeal to a distinction between explicit and implicit conception, direct or indirect, full or weakly oblique. In a full sense a propositions is conceived in a full sense only when it is entertained in full content and with genuine understanding. In a far weaker sense, a proposition may be conceived *of* in any of a number of indirect ways—as the core propositions that a speaker will be arguing for, for example, but that I have not yet heard. We can thus think of the numerical argument as leading us to the weaker conception of propositions that are beyond conceivability in the full sense.

We can perhaps press the paradoxical character of the argument, however, by explicitly considering all facts that might be conceived *of*. In a similar fashion, we might consider all the facts that might be referred to in any way, either directly or obliquely. Given the basic Cantorian argument, there will be more facts than can be conceived of, and more facts that can be referrred to in any way. If the 'however possible' defines a fixed set, there will be specific facts that cannot even be conceived *of*, and which cannot be referred to in any way. But have we not just conceived of those? Have we not just referred to them?

There is an escape clause here that we will return to below and that we will in fact use as a window to the world beyond. For now let us note that the core argument, like its predecessors, relies on essential assumptions of number: the assumption of fixed collectivities with a given cardinality.

Applied to conceivable or referrable facts, the argument take the same form as that used earlier to show that the truths within any language will be outstripped by the facts of a world beyond. In that case the character of languages does indeed commit us to a fixed collectivity of expressions and thus of expressible truths that has a specific cardinality. When it comes to facts conceivable in any sense, to facts referrable in any sense, or to all the facts themselves, it will be these assumptions of collectivities bound by familiar principle of number that we will have to leave behind.

Our proposal is that we not treat the reasoning that leads us to such a point as somehow illegitimate, in need of a 'solution.' Our proposal is that we let the logic lead us to a genuine though radically unfamiliar realm beyond.

5. Facing Facts

Any world of fact must extend beyond language and beyond explanation. In at least some sense, it must extend beyond conceivability as well. Is any glimpse of the character of such a world simply impossible?

We think not. Our goal is to offer a glimpse of that world beyond.

The results that have led us here should warn us that the full world of fact will not be conceived in standard terms. Some of our familiar ways of approaching things must be compromised. Interestingly, they may be compromised in any of several ways.

If there is a world of fact, we will propose, its collectivity must be conceived as a *plenum*. Plena are supra-numerical collectivities that violate at least one of several standard logical assumptions. Among such supra-numerical collectivities are the totality of all things, of all abstract objects, of all propositions. Like these, we propose, the world of fact constitutes a plenum.

A. Consider a Cantorian argument applied directly to the totality of facts. Given any such totality, there will be more sub-collections of the totality than there are members. But for each of those sub-collections there will be a distinct fact: that a given fact f is or is nor a member of that sub-collection, for example. There will then be more facts than contained in the totality of facts.

Something has to give. The argument can be perspicuously rendered as an aporetic triad:

1. The Cantorian assumption: There will be more sub-collectivities of any collectivity than there are members of that collectivity.

2. The Factual assumption: For any sub-collectivity of any collectivity there will be a distinct fact.

3. The Totality assumption: there is a collectivity that contains all facts.

Given (1), there will be more sub-collectivities of the collectivity assumed in (3) than there are members of (3). Given (2), there will be more facts than there are members of (3). Given (3), there will be more facts than are contained in a collectivity that contains all facts.

In this form the aporia is clearly one of number: any supposed totality of facts will have more members than it has members. Whatever *number* it contains, it must contain more than that number. Our exploration will involve digging beneath that concept of number. We begin, however, by surveying possible options.

One option is to deny (3). Despite appearances, despite deep intuitions, and perhaps despite our apparent ability to quantify over facts in general, there simply is no totality of facts. The world of facts is essentially incomplete: facts refuse to form a whole. The universe, on such an approach, is incomplete. It is this option that one of us has argued for in earlier work.[4] Aristotle, Kant, and Russell can be seen as precursors.[5] 'Indefinite extensibility' approaches, in denying a completed totality, can also be seen in this tradition.[6]

Another option is to deny (2). Despite appearances and despite deep intuitions there are things regarding which there are no facts. The things are there, they are what they are, but there is no fact regarding them. However difficult to believe, such an approach has also been attempted.[7]

The third option, which we will pursue, is to deny (1). There are collectivities for which Cantorian assumptions do not hold: collectivities beyond standard principles of number.

These collectivities will in fact be *defined* as having a unique member for each of their sub-collectivities. For any conception of their contents at any moment of thought—for any snapshot of membership at any conceptual moment—these collectivities will contain more. These collectivities, beyond standard assumptions of either sets or any collectivities like them, are *plena*.

These collectivities will in fact be *defined* as having a unique member for

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[4] Grim, *The Incomplete Universe*.

[5] Graham Priest, *Beyond the Limits of Though* (New York: Oxford University Press 2002), 229.

[6] See Stewart Shapiro and Crispin Wright, "All Things Indefinitely Extensible," in *Absolute Generality*, eds. Agustín Rayo and Gabriel Uzquiano (Oxford: Oxford University Press, 2006), 255.

[7] Keith Simmons, "On An Argument Against Omniscience," American Philosophical Association, New Orleans, April 1989.

each of their sub-collectivities. For any conception of their contents at any moment of thought—for any snapshot of membership at any conceptual moment—these collectivities will contain more. These collectivities, beyond standard assumptions of either sets or any collectivities like them, are *plena*.

We can construct a graphic example if we think of patterns of one or more patches on a two-dimensional plane, where each patch of a pattern must have an area. A pattern in our sense consists of a collection of patches that need not be contiguous, and indeed that might overlap. Graphically portrayed, one might think of a patch within another patch distinguished by a different color. For completeness, we include a completely blank plane as a pattern as a well.

Given this concept of patterns, it is clear that both any sub-pattern of a pattern and any collectivity of patterns will themselves constitute a pattern. The totality of all patterns will constitute a plenum, since every collectivity of elements of that totality—analogous to the elements of the power set of a set—will also constitute an element of the totality.

If propositions are understood as claims to facticity in the abstract, beyond any linguistic limits of mere statements, the totality of all propositions will constitute a plenum. For every collectivity of propositions there will be a distinct proposition—that a favored proposition p is included in that collectivity, for example (whether true or not)—and thus the totality of propositions will contain as many propositions as there are collections of propositions. The totality of things will constitute a plenum, if 'things' is broad enough to include collections. Every collectivity of things will constitute a thing in its own right. The totality of abstract objects will constitute a plenum for similar reasons.

Moreover, facts taken as a whole will form a plenum as well. There indeed is a world beyond language, sets, and systems. This, to be specific, is the plenum constituted by the world of facts.

There are several approaches to the aporetic triad that have points in common with the approach we take here, though we regard these as mere points of contact, short of the full metaphysical vision of a trans-numeric world of fact that we propose. In an attempt to understand truth, Hans Herzberger, Anil Gupta, and Nuel Belnap envisage truth as a concept that forces its own revision, much in the way that any attempt to conceive of the contents of a plenum as a fixed collectivity forces a revised vision of its further extent.[8] In the same light, an approach in terms of 'indefinite extensibility' has points of contact with our own. Graham Priest urges us to welcome any inconsistency in the aporetic triad for its

[8] Hans Herzberger, "Notes on Naïve Semantics," *Journal of Philosophical Logic* 11 (1982): 61, Anil Gupta and Nuel Belnap, *The Revsion Theory of Truth* (Cambridge, MA: MIT Press, 1993).

own sake, opening dialethic arms to 'true contradictions.'[9] We will not knowingly embrace contradiction. There is nonetheless a way of reading some of Priest's conclusions—that totalities at issue are both complete and not—that does resonate to some extent with the vision of plena we wish to present.

B. As expressed above, the aporetic triad turns on a concept of number that is buried within the Cantorian assumption. 'There will be *more* sub-collectivities of any collectivity than there are members of that collectivity.' On a Cantorian conception of number, the claim that a collectivity Y contains more than another collectivity X means simply that any line-up of the two such that every member of X is assigned a distinct member of Y will leave out some member of Y: the 'more' that Y contains.

Cantor's theorem is that the subsets of any set S—elements of its power set **P**S—will necessarily outnumber the elements of S. The proof is a proof that there can be no mapping M of elements of S onto distinct elements of **P**S that doesn't leave some element of **P**S out. For any proposed M, the proof offers a specific element of **P**S that must be left out. Here two points are of particular note. The first is that the 'specific element of **P**S' or subset of S that is necessarily excluded from the mapping M is itself specified in terms of M and a specific relation R. The second is that the 'necessary exclusion' of that element is exclusion on pain of contradiction. Derivation of the contradiction demands exclusive and exhaustive alternatives regarding an element of **P**S and that element of S mapped to it by M. S must either stand in relation R to its corresponding element or not. At its foundations, then, the 'more' of our aporetic triad is a matter of contradiction given exclusive alternatives and a peculiar reflexivity involving a mapping M and relation R.[10]

Although our target is collectivities well beyond mere sets, it is worthwhile to review the general mechanisms of the familiar set-theoretic proof. We assume any mapping M designed to assign each member of S to a unique member of its power set **P**S. The relationship R is set-membership, a crisp binary relationships fully obtaining or failing to obtain between any two candidates. We then consider a particular subset of our original set, specified in terms of M and R: the set D (for diagonal) of precisely those members of S which are not members of the subset to which they are assigned by our mapping M. If M fulfilled the conditions of 'same number,' giving us a one-to-one correspondence onto all elements of **P**S, it would

[9] Priest, *Beyond the Limits of Thought*.

[10] Patrick Grim and Nicholas Rescher, *Reflexivity: From Paradox to Consciousness* (Frankfurt: Ontos Verlag 2012).

assign some member s of our original set to D. But given either of two exclusive and exhaustive alternatives regarding membership, any such assignment leads to contradiction. If s of S is a member of D, it will by specification of D *not* be a member: D is to contain *only* those elements of S that are not members of the subset assigned by M. If s is not a member of D, it will by specification of D be a member of D: D is to contain *all* those elements of S that are not members of their corresponding subset.

The power set of any set must be larger than the set itself. In the context of classical set theory, the obvious next question has always been 'and what of the set of all sets?' By virtue of containing all sets, it must contain the elements of its own power set. But won't we then be forced to conclude that it is larger than itself?

With an eye to possible exportation to the aporia regarding all facts, consider standard responses to the strictly set-theoretic issue of a set of all sets. The standard line, despite appearances, despite intuitions, and perhaps despite our apparent ability to quantify over sets in general, is to deny the existence of a set of all sets. One move here, kicking the problem upstairs, is to create a new department of 'classes,' to one of which all sets (but of course not all classes) are assigned.[11] Another move is to deny or restrict the power set axiom, required in standard axiomatization to give us **P**S for arbitrary sets S to begin with.[12] A third move, echoing a theory of types, is to attempt to restrict the specifications of subsets so as to exclude the specification required to give us D.

C. None of the standard options for dealing with a set of all sets can be said to be intuitive. All look like cheating. All carry an atmosphere of the ad hoc. Parallels to those options become even less intuitive when we attempt to export them to the issue of a totality of facts.

For every collectivity of facts there will be a distinct fact: that a chosen fact is an element of that collectivity, for example, or that it is not. That a chosen fact is entailed by the collectivity, or that it is not. That the collectivity is finite, for example, or that it is not. That some of its elements entail other elements, or that all elements of that collectivity are logically distinct. Consider any of these 'collectivity facts' regarding the facts of a specific collectivity.

Consider now (a) the elements of a collectivity of all facts and (b) facts regarding collectivities of these, of any of the forms above: facts as to the facts they contain, facts regarding the facts they entail, the finitude or infinitude of the

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[11] The further sorrows of class theory are documented in Grim, *The Incomplete Universe* and Priest, *Beyond the Limits of Thought*.

[12] Christopher Menzel, "On Set Theoretic Possible Worlds," *Analysis* 46, no 2 (1986): 68. See also Menzel, "Sets and Worlds Again," *Analysis* 72, 2 (2012): 304.

collectivities at issue, or the like. We can think of the facts falling within the collectivity of a collectivity fact (b) as facts within its domain. Somewhat more informally, but to the same point, we might think of the facts within the domain of a collectivity fact as facts it is *about*.[13]

Take any one-to-one mapping M from the facts of (a) to the collectivity facts of (b). Any such mapping must leave some element of (b) out. Consider in particular all those facts on the left that do not fall within the domain of their associated collectivity fact. There will be a fact df about precisely that collectivity: that it entails a chosen fact f or that it does not, that it is finite or infinite, and the like. But there can be no element f* of (a) mapped to fact df. If f* falls within the domain of df, it cannot, by specification of df in terms of our mapping M. If f* does not fall within the domain of df, it must, again by specification of df.

In the context of the argument targeted to facts, the option of denying the existence of a set of all sets would be paralleled by a denial of any totality of all facts: denial of (3) in our aporetic triad above. On that line there is no world of all facts: the factual world refuses to form a coherent whole.[14] Such a route seems to violate the concept of a world.

The option of avoiding a set-theoretic diagonal set D by denying all sets within a power set **P**S is can be paralleled here by avoiding df, denying that any collectivity of facts is something about which there will be a fact. This amounts to a denial of (2) above. This route seems to violate the very concept of facts.

Neither of these options allows us a world of facts. One offers us a totality of something short of the ubiquity of facts. One offers us facts without a totality. On either approach, on pain of contradiction, we are again forced to conclude that there are too 'many' Cantorian facts to form a world.

One might choose simply to revel in contradiction. We take the result more seriously than that, as an invitation to explore a realm beyond. In the present line of inquiry we assume a genuine world of fact. We ask what results such as these have to show us about the possible character of that world, however strange.

What we explore is what must follow if we deny (1) of the aporetic triad. The world of facts, we propose, lies beyond a number of the Cantorian

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[13] The difficulties of pinning down the concept of aboutness in even the context of linguistic statements, making free use of the concept of designating expressions, became evident long ago in an exchange between Rescher and Goodman (Goodman, "About," *Mind* 70 (1961): 1, Rescher, "A Note on 'About'," *Mind* 72 (1963): 268). The current deliberations extend beyond language, targeting a relation of aboutness between facts and facts. In the context of facts, we'll argue, the concept of aboutness is not merely difficult to define but indeterminate in application.

[14] As in Grim, *The Incomplete Universe*.

assumptions. The world of facts forms a plenum.

6. The World of Fact as Plenum

We define a *plenum* as a collectivity that contains distinct elements corresponding to each of its sub-collectivities, where sub-collectivities follow the same pattern as subsets: something qualities as a sub-collectivity of a collectivity C just in case each of its members is a member of C.

In a membership plenum, such as a collectivity of all collectivities A, each sub-collectivity is itself a member of A. In other plena, such as the collectivity of all facts F, there is a fact regarding each sub-collectivity of F that is itself a member of F. Membership plena contain their own power collectivities. Other forms of plena contain members that map onto their power collectivities.

We take such plena to exist, with the world of fact as an example so intuitive as to be undeniable. The question for us, then, is not whether there is a world of fact but what such a world must be like.

A. We assume both (2) and (3) of the aporetic triad above. For anything that exists—and thus for any sub-collectivity of any collectivity—there will be a distinct fact. There is moreover a world of all facts. What we must deny, then, is the Cantorian core in (1): the claim that there will be more sub-collectivities of any collectivity than there are members of that collectivity.

The key to the Cantorian argument is that crucial concept of number: the claim that there will be *more* sub-collectivities of any collectivity than there are members of that collectivity. That 'more' amounts to the thesis that there can be no one-to-one mapping M from elements of a collectivity C to elements of its power-collectivity **P**C or some collectivity F**P**C which contains distinct members for each element of **P**C.

If we are to embrace plena as collectivities with members for each sub-collectivity, we must deny that there will be 'more' of the latter. We must hold that there **P**C be a mapping M from C to **P**C or F**P**C which leaves no element of the latter out.

In doing so we have to find the loophole in the Cantorian argument that attempts to show there can be no such M. That argument rests on specification of a particular element D of **P**C or F**P**C which stands in relation R to all and only those elements of C to which their corresponding M-correlate does *not* stand in relation R. Our assumed mapping, in assigning an element of C to every element of **P**C or F**P**C, must assign an element d to D.

B. The crucial step in the argument is the dilemma step. Does d stand in relation R to D, or not? If not, by specification of D in terms of M, d must stand in relation R to D. But if it does, again by specification of D, it cannot.

The lesson, we believe, is that for any plenum there will be inherent indeterminacy in R. For any M, any R, and any D definable in terms of M and R, the M-correlate to that D neither will nor will not stand in relation R to D. In the case of a simple membership plenum C, for every way M of assigning elements of C to elements **P**C one-to-one, the element d of the plenum assigned to that D by M neither will nor will not be a member of D. In at least some cases, the Law of Excluded Middle LEM will fail for the membership relation within plena. For some items x within a plenum P, it will be neither the case that $x \in P$ nor $x \notin P$. In that sense, some of the borders of plena will be imperfect, imprecise, or indeterminate.[15]

The lesson regarding a world of all facts is clear as well. The Cantorian argument regarding facts relies on 'collectivity facts': facts regarding whether a specific collectivity of facts contains or entails a specific fact, for example, or is finite or infinite. The crucial question of that argument is whether a specific fact lies within the domain of such a fact: somewhat informally, whether it is one of the facts that collectivity fact is about. Because the world of facts is a plenum, the relevant relationship—that a fact lies within the domain of another, is one of the facts it is about, or is one of the facts for which the collectivity fact holds—must in at least some cases be indeterminate. It is not always the case that a fact is either an element of a specified collectivity of facts or is not. It is not always the case that a fact is either one of the facts another fact is true of or is not. It is not always the case that one fact subsumes another, or is about another, or is not.

That, we suggest, is the lesson to be drawn from the clear existence of a world of fact. Given a total world of fact, various facts about the world will have to be indefinite, indeterminist, or undefined. Corresponding to a multitude of collectivity-defining characteristics Y there will be a multitude of factual theses of the form 'It is not always the case—it is not always itself a fact—that a particular fact f is either Y or not Y. What might be called alethic indeterminacy—indeterminacy of fact—will pervade the world of fact.

Our reflections have brought us to alethic indeterminacy from consideration of a fact's membership in a given collectivity of facts, or having a characteristic shared by certain facts. In that train of thought, it appears to be on the meta-level of facts about facts that is crucial. At this point both the substance and form of the result are reminiscent of Gödel, though with an enlarged perspective. Gödel

[15] This indeterminism bespeaks a curious parallelism between the ream of the theoretically very large—plena—and the physically very small—quanta.

showed that any consistent systematization of arithmetic will be incomplete, leaving the provable truth or falsity of certain arithmetical truths undetermined. The proof involves the technique of Gödel numbering, allowing statements of the base language to correspond to or 'encode' second-order statements regarding theoremhood within the system. Our conclusion also involves reflexivity, though it applies in the metaphysical realm well beyond logical systems: any totalization of fact is going to leave the status of certain factuality-claims indeterminate. Given the structural similarities, resonant results in this enlargement of perspective should perhaps not be entirely surprising.[16]

It should be emphasized that the denial of LEM at issue throughout is a *strong* denial, rather than invocation of either a third alternative or any number of additional alternatives. Were we to think in terms of three exhaustive categories—that a fact (i) falls within the domain of another, (ii) oes not, or (iii) neither does nor does not—we could construct a relation R in terms of teh second two that would be sufficient for resurrection of the basic argument. Were there *any* totality of exhaustive categories, we could do precisely the same. The strong denial of LEM is a denial that there is *any* set of exhaustive categories regarding the relationships between facts and collectivity facts at issue.[17] The lesson to be drawn from the clear existence of a world of fact is that a prime characteristic of some facts—that they take other as part of their subject collectivity—does not hold in terms of any set of exhaustive categories regarding all pairs of facts. In that sense, the lesson of a world of fact is that certain characteristics of facts themselves are not what we might have taken them to be.

The argument may well generalize to other characteristics of facts. It is worthy of note, however that it will not generalize to all. The Cantorian argument cannot be plausibly constructed in terms of just any relation R.

Consider an attempt to construct the argument in terms of logical entailment, for example. Some facts and some sets of facts logically entail others. For any M from facts to elements of the power set of a set of all facts, we might then envisage D as all those facts which are not entailed by the elements of the power set to which M assigns them. M must assign a fact d to that D.

But what then is the crucial question required for a Cantorian dilemma? We might first phrase the question as one of membership: Will d be a member of D or not? If it *is* a member, it will not be entailed by its corresponding set D. Interestingly, we cannot maintain that option: if d is a member of D, D certainly will entail d. But we can maintain that d is *not* a member of D. It follows that D

[16] See also Grim and Rescher, *Reflexivity*.

[17] See Rescher and Grim, *Beyond Sets*, chapter 6.

will entail d without containing it, but that does not give us contradiction. A set of propositions may entail many that it does not strictly contain.

We might alternatively ask whether d will be logically entailed by D. If it is not, it is an element of C not entailed by its M-correlate, and so will be a member of D. But as a member of D, of course, it will be logically entailed by D. The hypothesis that d will not be logically entailed by D is inconsistent. But the hypothesis that d *will* be logically entailed by D is not. In that case d, though not a member of D, will be entailed by D. Once again, a set of propositions may entail many that it does not strictly contain.

Given a world of facts, some relations—whether one fact falls within the collectivity addressed by another, for example—must be indeterminate. Logical entailment, on the other hand, need not be.

Though short of contradiction, there is a strange consequence of the argument phrased in terms of logical entailment. Because it can be run for any proposed one-to-one correspondence M from facts to collectivities of facts, the Diagonal construction D for *every* such M will entail whatever d is assigned to it.

C. We have defined plena as collectivities which take as members either their own subsets or elements such as facts mapped onto their subsets. Any world of fact would necessarily meet that criterion.

There are, we think, four options regarding plena:

1. Using standard logical principles, we might insist on Cantorian grounds that plena do not and cannot exist.

2. We might hold that plena do exist, but that the law of excluded middle fails to hold for all cases membership and crucial relations R.

3. We might hold that they do exist, but that the law of non-contradiction NC fails to hold in all cases for membership and crucial relations R.

4. We might hold that plena do exist, with every element of their power set as or corresponding to a member, and with power sets that are indeed larger than they are.

On the assumption of a world of all facts, (1) must be rejected. We have outlined (2) as a favored option, tracking some of its implications for the nature of facts. We consider (3) and (4) more radical options, but include consideration of these as well.

D. The dilemma at the core of the Cantorian argument takes the form 'Does d stand in relation R to D or not?' That dilemma assumes that its options are exhaustive—precisely the assumption denied in putting aside the law of excluded middle for such a case. That dilemma also assumes, however, that its options and their consequences are exclusive: that something cannot both stand in relation R to D *and* not. The force of the argument can be broken at that point if we simply shrug and accept both options.

The implication would be that for plena, issues of membership can be both 'yes' and 'no': in some cases collectivity c can both be a member of another collectivity c' and not be a member. In some cases a fact f can both fall within the domain of another fact f' and not fall within that domain.

Here consequences are roughly the dual of those outlined above. On denial of LEM, membership and whether a fact is among those another fact applies to are indeterminate in some cases. On a denial of the law of non-contradiction, these will be overdeterminate in some cases. In one case it is exhaustiveness of alternatives that is denied—that a fact is either among the collectivity to which another applies or that it is not. In another case it is exclusiveness of alternatives that is denied—that a fact cannot be both.

Our tendency, as noted, is to go for indeterminacy and the LEM. Another tack, however, would be to derive a disjunctive lesson. For plena, membership must either be indeterminate or overdeterminate in some cases. For facts, whether one fact falls within the domain of another must be either indeterminate or overdeterminate in some cases.

E. A last option, though the most radical, also has its attractions. Could there be a one-to-one mapping from a plenum to its subsets? From facts to sub-collectivities of facts? The answer from (2) and (3) is that there could be such a mapping. Plena need not be larger than themselves.

The last option is to accept the conclusion of the Cantorian argument. There can be no exhaustive mapping from a plenum to its subsets. Its power set is larger than it is, in that sense. But every one of its subsets appears as a member. It is therefore larger than itself. On this approach we maintain both the law of non-contradiction and the law of excluded middle. All the assumptions of the Cantorian argument stand, as does its conclusion.

Such an approach has some aesthetically pleasing elements. The idea that plena will be larger than themselves has an intuitive resonance with feelings one gets when thinking about a totality of fact, for example: having thought one had them all, one finds they are more. Plena seem to expand under our gaze.

There is also something pleasing in thinking of plena as the third step in size conception of collectivities. Finite sets are collectivities such that all proper sub-collectivities are smaller than the collectivity itself. Infinite collectivities are those such that some proper sub-collectivities are as large as the collectivity itself. Plena are collectivities such that some proper sub-collectivities are larger than the collectivity itself.

There is however a major sacrifice here as well. On such an approach there will be no one-to-one mapping from plena onto themselves. If there were, there would be a mapping onto their power set, violating the conclusion of the Cantorian argument.

The non-existence of a one-to-one mapping for plena would mean that there is no relation that holds one-to-one between members of a plenum. That would seem to force us to the most radically contentious option of all: to hold that items of a plenum will even fail to map onto themselves by way of a relation of identity. Even self-identity will fail for at least some items of a plenum.

For collectivities, this would appear to mean that whether something is identical to another—is the same collectivity as another—would in some cases be indeterminate. For facts, this would mean that whether something is the same fact as another would be indeterminate. On such a view we would have individual facts, we would have a totality of all facts as a plenum, but the concept of 'the same fact' would lose its grip. Here perhaps is the most complete sense in which we would lose the concept of number: we would lose the concept of distinct entities involved in the counting.

We cannot say that we recommend such a route: after all, "everything is what it is, and not another thing." Were one to take such an approach, however, we think the appropriate route would be to emphasize the extent to which the concept of identity in general becomes problematic at this juncture. Classically, identity is detailed in terms of features or properties: $x = y$ for $(\forall F)(Fx \equiv Fy)$. If having certain properties itself becomes problematic for elements of plena, the applicability of identity so understood may become problematic as well. It should also be noted that such a route, however radically contentious, is not without precedent: Peirce denies identity for elements of a continuum, which has a number of points of contact with plena as considered here.[18]

F. With the concept of plena in hand, we can return to some of the issues raised in previous sections.

[18] See Wayne C. Myrvold, "Peirce on Cantor's Paradox and the Continuum," *Transactions of the Charles S. Peirce Society* 30, 3 (1995): 508, Fernando Zalamea, *Peirce's Logic of Continuity* (Boston, MA: Docent Press, 2012), Benjamin Lee Buckley, *The Continuity Debate: Dedekind, Cantor, du Bois-Raymond, and Peirce on Continuity and Infinitesimals* (Boston, MA: Docent Press, 2012).

It is clear by Cantorian argument that there will be more facts than there are propositions conceived of in the course of human history. There are more sets of those propositions than there are those propositions themselves. But for each such set there will be a distinct fact. The world of fact will outstrip the world of human conception.

That alone may not seem surprising. Extending the argument in section III, however, seemed to lead us into paradox. A Cantorian argument can be run not merely on all proposition that have or will be conceived, but the propositions it is in any way possible to conceive: a collectivity of all *conceivable* propositions. On such an argument it appears that there will be propositions that cannot in any way be conceived. But does not our grasp of the argument itself demonstrate that we have in some way conceived of them? Similar paradoxes accompany Cantorian arguments regarding all facts that might be referred to in any way, either explicitly or indirectly. There will be more facts than these; but are we not at this point referring to those facts supposedly beyond reference?

In section III we alluded to an escape clause, pointing out that each of these relies on the essential assumption of fixed collectivities with a given cardinality. That assumption is the Cantorian assumpton (1) that we have abandoned in favor of plena in exploring a world of facts.

An escape from paradox by way of a similar denial seems called for in these cases as well.

These apparent paradoxes, we propose, like the question of a totality of facts, point to the existence of plena. The realm of conceivable propositions, conceivable facts, and facts to which we might at least obliquely refer may all form plena: collectivities for which every sub-collectivity corresponds to a member. If all of this holds for actual facts, it will clearly hold for the still richer realm of possibilities: these will all the more emphatically constitute a plenum.

On the assumption of such plena, cashed out in any of the ways we've outlined—by strong denial of the law of excluded middle, exceptions to the law of non-contradiction, or a vagueness of identity—the Cantorian argument falls short. When broadly construed so as to include oblique conception and reference, we can see the realm of possible reference and conception—like the world of fact itself—as forming a plenum.

We should remind ourselves that in a familiar range of more restricted considerations all the classical principles can still be maintained. It is only when we reach for a grasp of totalities such as the world of all facts that we turn the page, forcing us to resort to new devices. Are compromises in familiar principles such as the law of excluded middle too high a price to pay for recognizing the existence of plena? Here the simplest answer, we think, is that we have no choice: it seems inescapable that there must be a world of fact as a whole. If so, here as elsewhere, it is our thinking we must mold to the world rather than the other way around.

Newtonian physicists confronted modern science with a physically infinite astronomical cosmos the contemplation of whose vastness filled Pascal with vertiginous fright. Cantorian set theory confronted modern mathematics with a qualitatively infinite numerical realm of numberless quantities.

The present deliberations confront modern philosophy with an epistemically infinite manifold of fact. Modernity is replete with challenges of coming to terms with the many guises of infinitude. Our discussion here is simply another instance of this larger phenomenon.

7. Conclusion

It is clearly demonstrable, from a number of sources and in a number of ways, that we face major limitations in the face of a world beyond the accustomed horizons of thought.[19] Our axiomatics imposes limits on formalization, with corresponding limits on explanation and the principle of sufficient reason. Godelian arguments show that demonstrable fact cannot exhaust fact.

Our language imposes limits on expressibility, limits that extend even to all possible languages. We argue that even expressible fact cannot exhaust fact. Beyond these, even conceivability faces inherent limits: the world of facts necessarily outstrips the world as we conceive it.

Despite those limitations, we propose that we can get a glimpse of the world of fact beyond. We can limn its general shape as that of a plenum: a collectivity that includes elements corresponding to all sub-collectivities.

Recognition of that fact, however, also forces us to recognize that such a world is unfamiliar in at least one of several ways. There is indeed a world of fact. But certain relations of facts to facts that might be assumed unproblematic—such as the question of whether one fact falls in the subject domain of another--will have weaker logical properties than we might have assumed. We have to conclude that whether one fact is about another may be indeterminate, in the sense of a strong denial of the law of excluded middle, or over-determinate, in the sense of a violation of the law of non-contradiction. A third alternative is that both of these hold, but hold for a range of things that are themselves less determinate that we

[19] Although such a phrase and much of the spirit of our piece echo Graham Priest's title for *Beyond the Limits of Thought*, it should be clear that his acceptance of contradictions is just one of the approaches we've outlined.

might have taken them to be. On the third alternative, it is a principle of identity that fails to hold in all cases: 'the same fact' loses its grip.

Language is a purposive instrument. Ordinary language has evolved for everyday use. Logico-Mathematical language primarily for logico-mathematical purposes. But beyond those familiar purposive horizons there lies the realm of abstract deliberation—a conceptual Wild West outside the pale of familiar logical law. Here the very questions one asks tend to be nonstandard. When you ask extra-ordinary questions, we propose, you must expect extra-ordinary answers. The reality beyond our conceptual horizons is a world about whose *being* we can reasonably say something but regarding whose n*ature* we do and can know effectively nothing. Our acknowledgment of this world is a constructive reminder to being honest and humble. It is the epistemic equivalent of the Roman functionary whose task was to give the emperor an ongoing reminder: "Remember that thou are but mortal."