THERE IS NO SET OF ALL TRUTHS

By PATRICK GRIM

AN important philosophical consequence of Cantor's work has apparently been overlooked. There can be no set of all truths.

I

The proof is as follows.

Suppose that there *is* a set of all truths $\mathscr{T}$:

$$\mathscr{T} = \{T_1, T_2, T_3, \ldots\},$$

and consider further all subsets of $\mathscr{T}$, elements of the power set $\mathscr{P}\mathscr{T}$:

$$\begin{array}{l} \phi \\ \{T_1\} \\ \{T_2\} \\ \{T_3\} \\ \vdots \\ \{T_1, T_2\} \end{array}$$

$$\begin{array}{l} \{T_1, T_3\} \\ \vdots \\ \{T_1, T_2, T_3\} \\ \vdots \end{array}$$

Now to each element of this power set will correspond a truth. To each element of the power set, for example, T_1 either will or will not belong as a member. In either case we will have a truth:

$$\begin{array}{l} T_1 \notin \phi \\ T_1 \in \{T_1\} \\ T_1 \notin \{T_2\} \\ T_1 \notin \{T_3\} \\ \quad \vdots \\ T_1 \in \{T_1, T_2\} \\ T_1 \in \{T_1, T_3\} \\ \quad \vdots \\ T_1 \in \{T_1, T_2, T_3\} \\ \quad \vdots \end{array}$$

There is of course nothing special about T_1 here — we could have used any particular truth in its place. There are also myriad other ways of constructing a distinct truth for each element of the power set $\mathscr{P}\mathscr{T}$.

To each element of the power set will correspond a distinct truth, and thus there will be at least as many truths as there are elements of the power set $\mathscr{P}\mathscr{T}$. But by Cantor's power set theorem the power set of any set will be *larger* than the original.[1] There will then be *more* truths than there are members of $\mathscr{T}$. Some truths must be left out, and thus $\mathscr{T}$ cannot, as assumed, be a set of *all* truths.

II

Let me mention just one application of the argument above, against a common approach to possible worlds.

Possible worlds are often introduced as maximal consistent sets of propositions — proposition-saturated sets to which no further proposition can be added without precipitating inconsistency — or as some sort of fleshed-out correlates to such sets.[2] The *actual*

[1] See for example Irving M. Copi, *Symbolic Logic*, fifth edition (New York: Macmillan, 1979), pp. 189–90.

[2] See for example Robert Merrihew Adams, 'Theories of Actuality', *Noûs*, 17 (1974), 211–31, and Alvin Plantinga's treatment of worlds in terms of books in *God, Freedom, and Evil* (Grand Rapids, Michigan: Wm. B. Eerdmans, 1980), pp. 35–44, and *The Nature of Necessity* (Oxford: Clarendon Press, 1974), pp. 44–69.

world, on such an account, is the maximal consistent set of propositions all members of which actually obtain – a maximal and consistent set of all *truths* – or is an appropriately fleshed-out correlate to such a set.

By the argument above, however, there can be no set of all truths. *Any* set of true propositions will leave some true proposition out, and thus there can be no maximal set of truths. Given this notion of possible worlds, then, there can be no *actual* world.[3]

III

The general argument above, of course, applies explicitly only against a *set* of all truths. It quite clearly relies, moreover, on a crucial assumption of bivalence regarding set membership.

We might then hope to dispel the air of paradox and to save a category of all truths by recourse to many-valued set theories or to the non-set *classes* of alternative set theories.

Here let me say simply that I am not sanguine about our prospects. Many-valued logics exhibit many-valued forms of the Liar and of Russell's paradox,[4] and my guess is that they will exhibit many-valued forms of the Cantorian argument above as well. Alternative set theories seem capable of including a universal class only at some unacceptable cost, such as crippling mathematical induction.[5] My guess is that the same may hold for any attempt to include even a *class* of all truths.

It might appear at first glance that there is a conflict between the Cantorian result above and Lindenbaum's Lemma, in terms of which we *can* construct maximal proof-theoretically consistent sets for familiar formal systems.[6] The conflict is merely apparent, however, since (for one thing) Lindenbaum's Lemma relies crucially on the fact that wffs of such systems are explicitly finite. No such limitation is imposed on the truths of $\mathscr{T}$ in the Cantorian argument.

Lindenbaum's Lemma can be seen, however, as preserving a notion of maximal proof-theoretically consistent sets for certain systems, and possible worlds construed in terms of them, as important tools for the logician. The possible worlds that the Cantorian result impugns are those grander entities, corresponding to sets of *all truths*, so tempting to the metaphysician.

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[3] For a similar argument against such an approach to possible worlds, using a variation on the Liar, see my 'Some Neglected Problems of Omniscience', *American Philosophical Quarterly*, 20 (1983), 265–76.

[4] See esp. Nicholas Rescher, *Many-Valued Logic* (New York: McGraw-Hill, 1969), pp. 87–90, 206–12.

[5] See esp. W. V. O. Quine, *Set Theory and Its Logic* (Cambridge, Mass.: Belknap, Harvard University Press, 1963), pp. 287–389.

[6] See for example Geoffrey Hunter, *Metalogic* (Berkeley: University of California Press, 1971), pp. 110–11 and 177–8, and Elliot Mendelson, *Introduction to Mathematical Logic* (Princeton: D. Van Nostrand, 1964), pp. 64–5 and 93.